Developments in Parrondo's Paradox

Derek Abbott

Abstract Parrondo's paradox is the well-known counterintuitive situation where individually losing strategies or deleterious effects can combine to win. In 1996, Parrondo's games were devised illustrating this effect for the first time in a simple coin tossing scenario. It turns out that, by analogy, Parrondo's original games are a discrete-time, discrete-space version of a flashing Brownian ratchet—this was later formally proven via discretization of the Fokker-Planck equation. Over the past ten years, a number of authors have pointed to the generality of Parrondian behavior, and many examples ranging from physics to population genetics have been reported. In its most general form, Parrondo's paradox can occur where there is a nonlinear interaction of random behavior with an asymmetry, and can be mathematically understood in terms of a convex linear combination. Many effects, where randomness plays a constructive role, such as stochastic resonance, volatility pumping, the Brazil nut paradox etc., can all be viewed as being in the class of Parrondian phenomena. We will briefly review the history of Parrondo's paradox, its recent developments, and its connection to related phenomena. In particular, we will review in detail a new form of Parrondo's paradox: the Allison mixture—this is where random sequences with zero autocorrelation can be randomly mixed, paradoxically producing a sequence with non-zero autocorrelation. The equations for the autocorrelation have been previously analytically derived, but, for the first time, we will now give a complete physical picture that explains this phenomenon where random mixing counterintuitively reduces randomness.

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1 Introduction

> *"Can the weaker be the stronger?"*
> Kwai Chang 'Grasshopper' Caine
> in "Chains" (Episode 9)
> *Kung Fu*, Season 1, 1972.

Parrondo's paradox is where losing situations combine in order to win, and is exemplified by simple coin tossing games [1] that readily yield to physical and mathematical exploration [2, 3]. The Parrondian paradigm is one of 'survival of the weakest' and is a counterintuitive nonlinear effect. Parrondo's original games comprise simple coin tossing and are thus not game theory in the von Neumann sense [4] where players make decisions—however, in their original form they can be thought of as game theory in the Blackwell sense [5], and more recently Behrends has extended Parrondo's original games to include player strategy [6,7] thus bringing them into the von Neumann realm. Consequently, in the following review we will use the term *game-theoretic* in its most inclusive sense—in the new field of quantum game theory, it is interesting to note that the phrase 'game theory' is also used broadly. In general, the emerging interest in game theory in the field of physics [8] uses the term in its broadest sense.

This Chapter is constructed as follows. Firstly, we take an entertaining look at a number of everyday examples of 'losing to win' or where the 'weakest is the strongest', to illustrate that the idea is widespread and to motivate the topic. Then we briefly go through the history of Parrondo's games, how they are constructed, how they work, and trace their origins to the flashing ratchet and the Feynman-Smoluchowski ratchet. We review these ratchet devices in order to help the reader develop an understanding of the physical origins of Parrondo's original games. We then review some recent developments in the study of Parrondo effects in a number of diverse fields and also review some interesting closely related phenomena. In particular, we show how *volatility pumping* on the stock market, in its simplest form, can be simulated and point out its similarities as a ratcheting effect.

Finally we conclude with a discussion on the thermodynamics of chance and then exploit thermodynamic analogies to develop a physical picture to explain a new intriguing Parrondo effect: the *Allison mixture* [9]. Here, the Allison mixture is the counterintuitive situation where the random shuffling of random sequences begins to 'erase' their randomness. In other words, two sequences that are incompressible can be randomly interleaved resulting in a sequence that has some compressibility.

2 The Ubiquity of 'Losing in Order to Win'

Is the Parrondian paradigm of losing to win that surprising? After all, many of us are familiar with the concept of a sacrifice in the game of chess. Also in biology, it is known that as a genotype evolves, the fitness landscape is usually not flat but can have a valley, i.e., fitness declines, before the genotype rises to a higher level of fitness (e.g. see [10]). It has been speculated that Winston Churchill deliberately turned a blind eye to the November 14th, 1940, German bombing raid of the city of Coventry [11]—the implication being that Churchill allowed the bombing to proceed to disguise the fact that the German Enigma code had been decrypted at Bletchley Park, in order to save more 'important' cities than Coventry. Whilst many historians now believe this to be an urban legend, the anecdote nonetheless illustrates the general notion of sustaining a loss in order to win. In the engineering literature it is known that individually unstable systems can become stable if coupled together [12] and in the physics literature we have the principle that many imperfect devices can be combined to produce a near-perfect device [13]. If one believes that the biogenesis of life occurred in a primordial soup, one is faced with the conundrum that a number of 'losing' effects must somehow have cooperated to produce life out of the incipient disorder [14].

These and the examples that are about to follow illustrate that the idea of sustaining losses in order to win is ubiquitous and thus prompts us to study the new game theory of 'losing to win', motivating the detailed study of Parrondo's paradox in order that we might understand the general principles more deeply.

2.1 The Trueling Problem

The *truel* is similar to a traditional duel except three, rather than two, players have a shoot out. The last man standing is the winner. Here we specifically discuss the *sequential* truel where the gunmen take it in turns to shoot. The detailed rules are in the caption of Figure 1, but essentially the weakest Player A has first shot, then Player B, and so on. If you are the weakest player and you start, what is your best strategy? Should you try to eliminate the strongest out of Player B or Player C? The answer surprisingly is neither! It turns out that your best strategy for survival is in fact to waste your bullet and shoot into the air. For the full analysis see Flitney *et al* [15] and Amengual *et al* [16]—however, the general principle is that by sacrificing your turn you leave a greater chance for the more powerful players to fight it out between them. This game is an intriguing example of where 'survival of the weakest' relies on making the weakest first move. The rules of the game presently assume unlimited resources—the case of limited rounds of bullets has been considered [17].

The concept of the truel has some bearing on the dynamics of political parties—in large democracies it is interesting to note that invariably there are always two major parties: something akin to 'Republican' and 'Democrat'. All other parties outside the main two tend to be very minor in comparison. Why is this? One possible conjecture is that as soon as a third party starts to become significant it takes more votes away from the politically closest major party than the diametrically opposite party. This becomes self-defeating, as then the diametrical opposition wins! So, it is far more strategic to either stand back and let the two major parties fight it out (rather like the truel), or collude and join with the politically closest party. A famous example of this effect was in the US 2000 election, where Bush won by a small margin (a margin so narrow that it may be considered as statistical noise)—however as much as 2.47% of the vote went to a minor party led by Nader. It is often speculated that had Nader not run, then Gore would have risen above the noise level.

Similarly, this type of reasoning has some explicative power for suggesting why in wars there are usually only two major factions: the 'enemy' and the 'allies'. There do not appear to be many major historical cases where there are simultaneously an Enemy 1 and an Enemy 2 that fight each other. That is because if one is the smaller of the three factions, it is far better to play the truel gambit by stepping back and letting the larger two enemies annihilate each other. It may also be interesting to extend this line of reasoning to explain why a marriage between two partners appears to be more stable than an *n*-partner marriage. Another interesting open question is that in sexual reproduction why are there only two sexes (male and female) and not *n* distinct sexes that mate either sequentially or simultaneously?—this is in fact a major field of research with rich multidisciplinary activity [18–22].

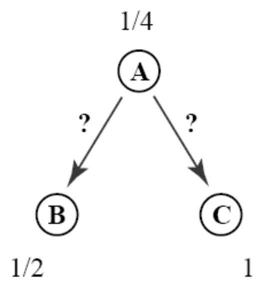

Fig. 1 The trueling problem. This is similar to a wild west dual, except that we now have three instead of two players. The weakest is Player A who only shoots with a success rate of 1/4. Player B is a better marksman who shoots with a success rate of 1/2, and Player C is a gunman who is a guaranteed perfect shot. The rule is that Player A shoots first, then Player B, then C, and so on in sequential order until one man is left standing. Each player must shoot on each turn, and you may assume unlimited resources, i.e. an infinite supply of ammunition. If you are Player A, what is your best starting strategy in order to survive? Should you shoot Player B or C? The solution is surprising, as it turns out that the weakest player can strengthen his chances by making the weakest move.

2.2 The Interplay of Redundancy and Pleiotropy

The term *pleiotropy* describes an agent that performs multiple tasks [23, 24], while redundancy is when multiple agents perform the same task. This is clearly illustrated in Figure 2 where we see that pleiotropy can be thought of as the inverse of redundancy. Pleiotropy and redundancy can be ubiquitously seen in many every day networks, ranging from neural interconnections through to client-server based networks made up of server nodes and client nodes.

Figure 2 shows that, individually, pleiotropy and redundancy are rather like 'losing games', as redundancy comes at high cost and pleiotropy comes with low robustness. Figure 3 illustrates that a mixture or interplay between pleiotropy and redundancy helps to overcome their individual disadvantages. Biological systems provide important examples of pleiotropy and redundancy [25,26]—intercellular messenger molecules such as cytokines may act as links between nodes (cells) [27]. A deeper knowledge of how pleiotropy and redundancy operate within the cytokine networks, may improve understanding of how to better manipulate disease states [28–30]. To date, little work has been carried out to explore the trade-offs between pleiotropy and redundancy in an evolutionary computational paradigm—future work in this area may help to explore the general principles behind such trade-off in the presence of both limited and unbounded resources. This may enable us to answer a number of fundamental open questions about how real biological, social, and electronic networks are optimally wired.

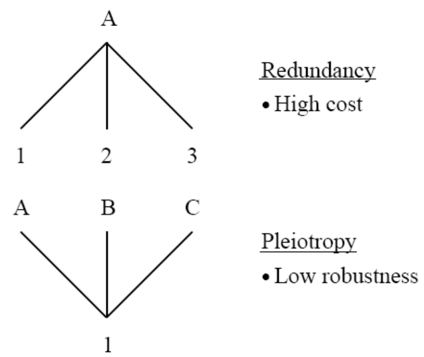

Fig. 2 Redundancy versus pleiotropy. In the top example, we have Agents 1, 2 and 3 all performing one Task A. This redundancy provides robustness at the cost of providing multiple agents. In the bottom example, we have Agent 1, performing three tasks A, B, and C. This pleiotropic situation provides the fulfillment of multiple tasks, but at the expense of low robustness—should Agent 1 fail to function, there would be a catastrophic reduction in output.

2.3 Costly Signalling

A large area of research where there is a complex interplay of both losing and winning strategies is that of *costly signalling* [31]. Costly signalling is a term used by evolutionary biologists for the situation whereby an animal advertises its fitness, for

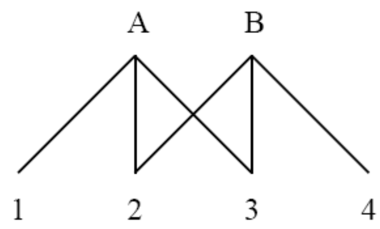

Fig. 3 Mixing redundancy and pleiotropy. Here we see the interplay of pleiotropy and redundancy can overcome their individual disadvantages shown in Fig. 2. We can intuitively see that we have increased robustness at lower average cost per output function.

example, for procuring a mate. In order to ensure that the signal is 'honest' it has been conjectured that it must come at a cost to the animal—otherwise it would be too easy to send out fake signals. The classic example is the fancy plumage of the male peacock. The larger these feathers are the more attractive the male becomes to his entourage of females. However, the feathers come at cost because (a) they make the male easier to spot by a predator, and (b) the feathers are cumbersome when escaping from a predator. Therefore, the conjecture is that the feathers are an honest signal, because they advertise that the male is fit enough to survive despite them. Thus in order to 'win' and find the optimal mate, the male plays the losing strategy of becoming vulnerable to predators.

3 History of Parrondo's Games

The original Parrondo games [1, 32] were devised in 1996 [33], as a pedagogical analogy of a flashing Brownian ratchet [34]. Since then they have stimulated research in diverse areas from economics [35], through to physical quantum systems [36–38], and population genetics [39–41]. For a more complete review, see [42].

Part of the original appeal of Parrondo's games is that they clearly illustrated effect of 'losing to win' for the first time with a toy model involving simple coin tossing games. Other related phenomena existed prior to Parrondo's games [34,43–49], but Parrondo was the first to show the effect in a clear game-theoretic form. His work was a landmark discovery because the simple analytical solution to his model enabled many workers to grasp the theory behind the phenomenon of 'losing to win.' Another significant event was when Parrondo's original games were first shown be formally related to the Fokker-Planck equation [50], then independently confirmed [51], and rigorously systemized [52]. This is significant as it opens up a formal link between thermodynamics and games of chance (see Section 5). Parrondo's games were originally inspired by the flashing Brownian ratchet [34, 42], and via the Fokker-Planck equation they are intrinsically related. The flashing ratchet was, in turn, inspired by the Feynman-Smoluchowski ratchet and pawl machine, which we now briefly review below.

3.1 The Ratchet and Pawl Machine

The ratchet and pawl machine is illustrated in Figure 4. The idea is that the ratchet wheel is biased to turn in one direction because of the action of a spring loaded latch (called the *pawl*). We see in Figure 4, that this ratchet wheel is connected to a vane via an axle. For generality, we can imagine that the vane is in a box maintained at a temperature T_1 and the ratchet is in a box at T_2. In 1900, the French Nobel Prize winner, Gabriel Lipmann was the first to do the thought experiment of shrinking this type of apparatus down to the scale of air molecules [53]. Could such a ratchet mechanism be used to rectify the random motion of molecules? Lipmann was the first to ask this question —it was a courageous question at that time considering that the discrete nature of matter had not yet been finally settled. Lipmann also asked if such rectification of random motion would violate the laws of thermodynamics. This caused a flurry of letters to journals, and finally in 1912 Smoluchowski came up with the canonical explanation that we hold to this day [53].

Smoluchowski correctly explained that the machine could legally rectify random motion and do useful work provided $T_1 > T_2$. To maintain such a temperature difference requires external energy. Thus the work output is at the expense of energy in—this principle universally holds for all types of engines and there is no violation of thermodynamics. However, the question is why does the machine stop working when there is no input energy (i.e. when $T_2 = T_1$)? Again, Smoluchowski brilliantly gave the correct answer—he explained that the pawl is also bombarded by air molecules and thus has a certain error rate of releasing the wheel to rotate in the wrong direction. He stated that at thermal equilibrium (i.e. when $T_2 = T_1$) we thus expect the probabilities of the wheel rotating either way to balance out, and therefore the machine can never do any net work.

However, Smoluchowski did not attempt to formally prove that the probabilities satisfied this required *detailed balance* condition. In 1963, Feynman was the first to attempt to do so using Boltzmann statistics showing the probabilities did indeed balance—however, he did not publish the fine details of the calculation [54]. Around 1980, I first attempted to derive Feynman's result from first principles and was unable to do it for 19 years. In 1997, I flew to Madrid to visit Parrondo and show him the problem—at the time we were unable to solve it, so we began discussing ratchets in general and Parrondo showed me his paradoxical games inspired by the flashing ratchet. From that meeting the seminal papers on Parrondo's games were born [1, 32]. Finally, in 1999 the problem of finding Feynman's detailed balance condition was cracked using level crossing statistics rather than Boltzmann statistics [55]—to this day the Boltzmann form remains an unsolved problem.

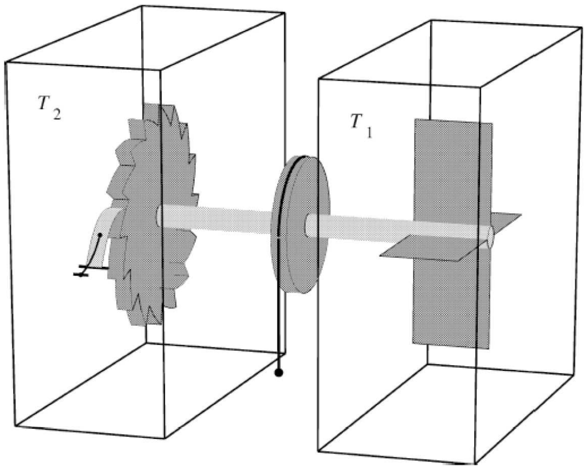

Fig. 4 The Feynman-Smoluchowski Engine (FSE). On the left is a ratchet wheel and a spring loaded pawl at temperature T_2. On the right is a vane at temperature T_1. The ratchet wheel is connected to the vane via an axle. Now imagine the whole device is small enough that the bombardment of air molecules causes the vane to rotate. As the whole machine is constrained to rotate in one direction, by the ratchet and pawl, the random motion of the air molecules is in-effect *rectified* to produce directed motion and the rotation of the axle can legally do useful work by lifting a weight via a small pulley. This is allowed provided that $T_1 > T_2$, whence we have net energy coming into the system to maintain this temperature gradient in the first place. Hence there is external energy driving the system to do work. However, when $T_2 = T_1$ there is no longer any net energy into the system, and it becomes therefore impossible to lift the weight. In the case of thermal equilibrium ($T_1 = T_2$) the spring loaded pawl fluctuates to release the wheel to go in the wrong direction. It turns out there is *detailed balance* and thus the weight jiggles up and down, but there is no net displacement on average. After [32].

3.2 'Kitchen Sink' Examples

We now briefly sample a few everyday (or what I call 'kitchen sink') examples of ratchets to illustrate their generality. In the previous section, the ratchet and pawl machine relied on the asymmetry of the ratchet teeth in order to operate—this is a spatial asymmetry, but ratchet effects are not limited to the spatial variable. Here we will see that an asymmetry in any arbitrary variable can lead to a ratcheting mechanism.

Every child knows that if you randomly jiggle a bowl of sugar, a bag of flour or a backet of sand, the lumps rise to the top—the scientific name for this phenomenon is the *Brazil nut paradox* [49], named after that fact that the large Brazil nuts rise to the top when you shake a bag of mixed nuts. Here, the random shaking of the container drives the large nuts 'uphill' against the gravitational gradient and thus this is clearly a Brownian ratchet—but where is the asymmetry? The asymmetry in this case lies in the size distribution of the particles and the fact that gravity is directional.

Another common example is that of *longshore drift* on a beach—here, it is common to find that the sand and shells tend to pile up on one end of the beach. This tends to happen when waves come in at an angle to the beachfront. So for example, if we have a south facing beach, and waves impinging in a north-east direction, then sand and shells will tend to pile up on the east side of the beach. Waves will come in a north-easterly direction, but ebb in a southerly direction, drawing out a ratchet-like profile, and dragging material toward the east. Incoming waves loosen the material, reducing frictional forces, and as the waves ebb away friction increases again. Thus the ratchet asymmetry is in the difference between angle of entry and angle of ebb, as well as difference in frictional forces experienced by the material.

When trading on the stock market, a common injunction is to *buy-low sell-high* in order to ratchet up one's gain. The asymmetry here is in price when we buy and sell, in order to exploit the natural price fluctuations in the market. When paying the restaurant check, at the end of a meal, a client will typically complain if he or she is over charged. However, if the check is accidentally under charged, the client might chose to stay silent. This asymmetry in the transmission of information is used the ratchet up the gain of the client. This is somewhat akin to the previous buy-low sell-high example.

So far we have seen spatial, frictional, informational, and money ratchets—but is a ratchet in the time variable possible? The answer is yes. To illustrate a time ratchet we briefly review the two-girlfriend paradox, which is an old chestnut due to Perelman [56] back in the 1930s, although it was later revived in the 1960s by Mosteller [57], and then modernized to be shown to be a ratchet in the 2000s [47]. The two-girlfriend problem is a brainteaser that goes as follows. Refering to Figure 5, we are told that Bill arrives at a train station at a random time each day. One train leaves for the east every 10 mins and one train leaves for the west every 10 mins—his strategy is to jump on whichever train arrives first. It turns out on average that he sees Monica nine times more often than Hillary. Why is this so? This seems a little hard to believe given that he arrives at a time random time each day. The answer is that this is a phase (time) ratchet and we must therefore look for an asymmetry in the time variable. In other words, there can be a phase difference between the trains. Imagine a scenario where the eastbound train leaves every 10 mins on the hour, and the westbound train leaves every 10 mins one minute later. If Bill arrives after, say, 10:11 am he will have a nine minute window that captures the eastbound train, but if he arrives after 10:10 am there is a one minute window in which the westbound train will arrive first. Thus if he arrives randomly, he is more likely to end up in the nine minute window, and thus sees Monica nine times more often. Table 1 summarizes the above examples highlighting the different forms of asymmetry we have identified.

Table 1 Everyday examples of Brownian ratchet effects. These examples demonstrate the general ubiquity of Brownian ratchets, where a bias toward a particular direction is a result of the interaction between random behavior and an asymmetry. The traditional focus has been on Brownian ratchets with asymmetry in a spatial variable—the right column shows that other types of variables can yield to asymmetric treatment, leading to directed motion.

Scenario	Source of Randomness	Asymmetry
Brazil nut paradox	Shaking the container	Particle sizes/Field
Longshore drift	Waves breaking on the beach	Geometry/Friction
Restaurant check	Waiter's error rate	Information
Buy-low, sell-high	Market fluctuations	Price
2-Girl paradox	Bill's arrival times	Train phase (time)

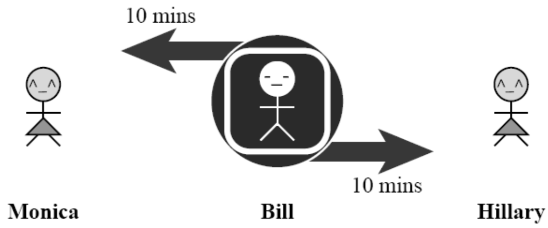

Fig. 5 The 2-girl paradox. The circle represents a city. At the center of the city is a train station, represented by a square. A westbound train leaves every 10 mins and an eastbound train leaves every 10 mins. Bill has two girlfriends, ones lives in the west and the other on the east side of the city. Bill arrives at the station at a random time each day and takes whichever train is there first. It turns out, on average, that Bill ends up visiting the westside girl nine times more than the eastside girl. Why? The solution to this puzzle reveals that the process is in fact a Brownian ratchet, where the asymmetry lies in the phase difference between trains.

3.3 The Flashing Ratchet

The principle of the Feynman-Smoluchowski Engine (FSE), in Figure 4, can be translated from a wheel to a linear mechanism. The flashing ratchet [34] is one example of this—we focus solely on this case as it is the type of Brownian ratchet that inspired Parrondo's games. The operation of the flashing ratchet is explained in the caption of Figure 6, were we see that particles can be 'pumped' uphill against a gradient by flashing the ratchet potential on and off. Energy that we input to toggle the potential on and off does work on the particles to move them uphill. The secret as to why the ratchet works is in the asymmetry of its sawtooth profile. It is this asymmetry that results in $P_{\text{fwd}} > P_{\text{bck}}$, giving rise to the net motion to the right in Figure 6. Parrondo's genius was in extrapolating from the flashing ratchet to coin tossing games. He visualized going uphill as gaining money, and the random position of a particle as being the accumulated capital. He recognized displacement along the flat potential, U_{flat}, could be simulated by winnings from a simple coin toss, say Game A, and that the gradient could be simulated by bias in the coin. He then recognized that displacement along the sawtooth potential, U_{saw}, could be simulated by winnings from a game composed of two coins, call it Game B. In this case it turns out that two coins are needed as each tooth is composed of two slopes—the longer slope pushes particles in the 'winning' direction and the shorter slope pushes them in the 'losing' direction. The periodicity of the sawtooth potential is simulated by choosing a selection rule for the coins based on modulo arithmetic. Switching between games A and B simulates the ratchet flashing on and off. In the following subsection, we now examine the construction of the games.

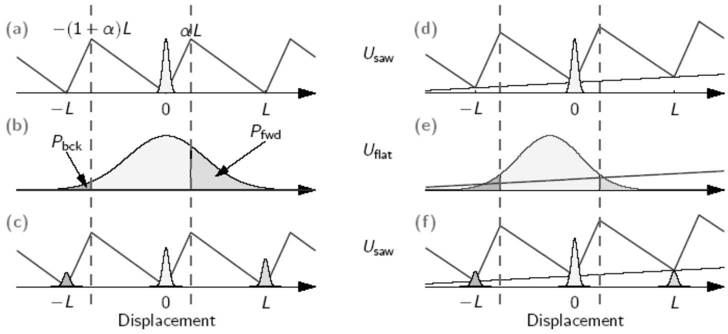

Fig. 6 The flashing ratchet of Adjari and Prost. (a) A ratchet sawtooth potential of pitch L. A Gaussian distribution of particles sits inside one of the potential valleys. (b) We now flash off the sawtooth potential off so that it becomes flat. The Gaussian distribution spreads as it is now unconstrained by a potential. Notice for convenience we have exaggerated the size of the Gaussian—in reality the area under the Gaussian is conserved. (c) We flash the ratchet potential back on. A rear tooth captures P_{bck} of the distribution, and a forward tooth captures P_{fwd}. A remarkable feature is that it turns out that this ratcheting procedure still operates when working against a gradient, as illustrated in (d)-(f). The flashing ratchet enables the particles to climb 'uphill' in a similar fashion to longshore drift on a beach. After [58].

3.4 Parrondo's Original Games

The key idea of Parrondo's games is that you can have two or more sets of games that are individually losing—however, if you periodically or randomly switch between the losing strategies, there are conditions under which it is possible to counterintuitively win. The games are constructed as indicated in Figure 7 to cleverly simulate the action of the flashing ratchet that was expounded in the previous subsection. Game A simulates the flat potential and Game B simulates the sawtooth potential. As we can see, in Figure 8, Game A and Game B are indeed losing games when played in isolation. Now, when we switch between the two games either periodically or randomly our winnings increase.

It has been shown using Discrete Time Markov Chain (DMTC) analysis [58] that the games are governed by very simple inequalities—Game A is losing provided,

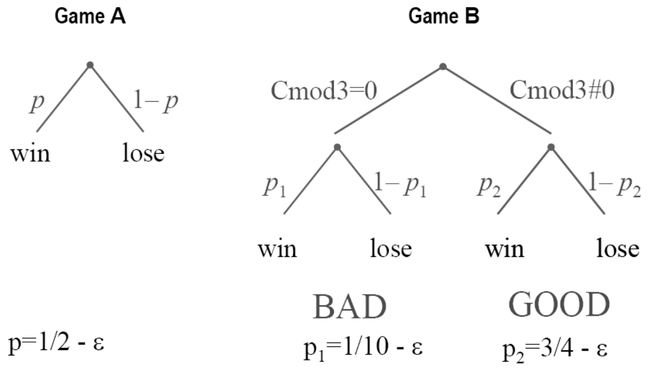

Fig. 7 The construction of Parrondo's original games. Game A is a simple coin toss that simulates the U_{flat} state of the flashing ratchet. The coin's bias is ε, which simulates the gradient of the flashing ratchet. Note that Game A is a losing game. Game B is composed of two coins. The 'good' coin is favorable and simulates the ratchet tooth's long slope and the 'bad' coin simulates the shorter slope of the ratchet tooth. For simplicity, your capital C goes up or down by \$1 every time you win or lose. You toss the bad coin if your capital is a multiple of three, otherwise you toss the good coin—this modulo arithmetic simulates the periodicity of the ratchet profile. The parameters of Game B are such that it is a losing game overall. When we switch periodically or randomly between the two losing games, surprisingly, we win.

$$\frac{1-p}{p} > 1 \tag{1}$$

and Game B is losing provided,

$$\frac{(1-p_1)(1-p_2)^2}{p_1 p_2^2} > 1 \tag{2}$$

and the random combination of Game A and Game B wins provided,

$$\frac{(1-q_1)(1-q_2)^2}{q_1 q_2^2} < 1 \tag{3}$$

where p, p_1, and p_2 are defined in Figure 7 and $q_1 = \gamma p + (1-\gamma)p_1$ and $q_2 = \gamma p + (1-\gamma)p_2$. Here, γ is the probability that Game A is selected and $1-\gamma$ is the probability of playing Game B. There are many ways to form a physical picture of why Parrondo's games work as they do—the picture becomes clearer once it is realized that Game A is coupled to Game B via the capital dependent rule. The first physical picture, due to Parrondo, is to simply to view the games as a discrete analogy to the flashing ratchet. An alternative picture is to see that Game B has a state dependence on capital that is forcing it to lose, and that Game A is acting as a source of noise that is breaking up that state dependence—this has been dubbed the Boston Interpretation, as it grew out of discussions at H. E. Stanley's group at Boston University [59]. In fact, it has been shown that as the amount of Game A

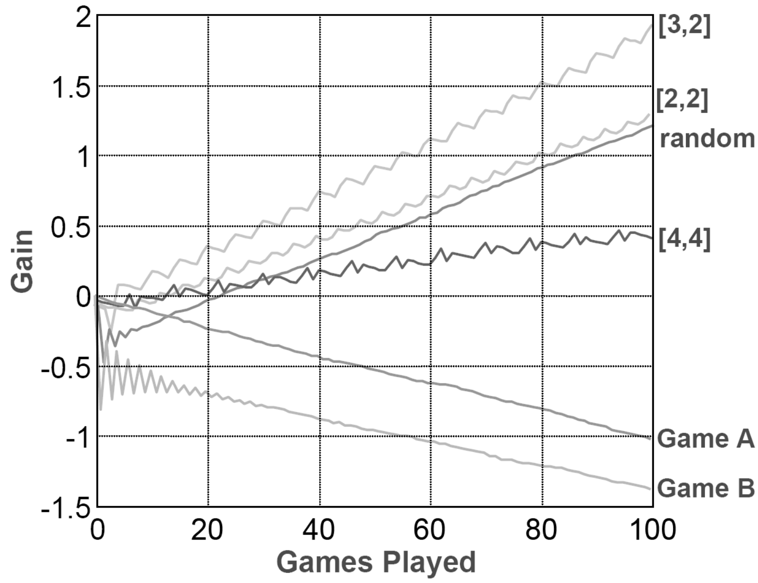

Fig. 8 The output of Parrondo's original games. The graph shows the amount of money gained versus the number of times we play. The parameters in this computer simulation are those listed in Figure 7 with $\varepsilon = 0.005$. As expected, we see that Game A and Game B are individually losing. If we play two rounds of A followed by two rounds of B, indicated as [2,2], we find that we remarkably win. In fact any periodic combination wins and the graph shows [3,2] and [4,4] as examples. The curve marked 'random' is that case where we decide to play A or B on the flip of an unbiased coin. The results are averaged over 50,000 trials to produce smooth curves, however, the trends are still observable for individual trials.

'noise' is gradually increased, by increasing γ, we find that the winnings follow a stochastic resonance-like curve [60]. Perhaps the most powerful interpretation is that of Moraal [61], who was the first to show that the games work due to a convex linear combination. This was later, and independently, reported by Costa *et al* [62], which is recommended as a clearer exposition for the new reader. The realization that a convex linear combination is at the heart of the games is a significant one, as it then more readily links Parrondo's paradox to control and optimization problems. Another key point in understanding why the games work is that the Inequalities 2 and 3 are nonlinear—Parrondo's paradox is essentially a nonlinear phenomenon. The fundamental reason why the governing inequalities are nonlinear is due to the state-dependence in Game B—in mathematical terms this is equivalent to saying that Game B is not a martingale. Finally, it should be noted that there are possibly an infinite number of ways to construct different nonlinear games that exhibit Parrondian effects. The open question is to search for the interesting cases that map onto physical and biological systems, and to investigate which display the largest regions of parameter space for the effect to occur. Progress has been made with developing differently constructed Parrondo games [63–65], but so far it is still early days and there is still much to explore in this regard.

4 Developments in Parrondo's Paradox and Related Phenomena

From an engineering viewpoint it is known that mixing unstable systems can result in stability, and the connection to Parrondo's paradox has been pointed out [12]—further investigation in this area may be of relevance to optimization problems. In the area of neural networks it is known that a network can perform better at *network generalization* if noise is added to the training data set [66]—this evokes the idea of losing in order to win and connections to Parrondo effects have yet to be investigated. Parrondo effects in spin systems [61] and quantum game theory [36–38], have been reported. Increasing our understanding of how to control decoherence is one of the motivating factors behind these developments in quantum game theory.

It has been shown [67] that Parrondo's original games can be rather elegantly described in terms Onsager rate equations [68, 69]—this suggests the possibility for future work in searching for chemical reactions that display Parrondo-like effects. Parrondo effects have also inspired work in the study of negative mobility phenomena [70], reliability theory [71], noise induced synchronization [72], spatial patterns via switching [73], and in controlling chaos [74, 75].

In the area of mathematics, Pinsky and Schuetzow [43] have shown that switching between two transient diffusion processes in random media can form a positive recurrent process—this can be viewed as a continuous-time version of Parrondo's games. It has also been shown that declining random branching processes can be combined to paradoxically increase [46].

The area of biology is still ripe with open questions for the study of Parrondo-like effects. There are many examples in biology of 'losing to win'—for example *sickle cell anemia* is deleterious and yet it can protect the host from contracting malaria. Conjectures have been mooted for the application of Parrondo's paradox to biogenesis [14], the dynamics of gene transcription in GCN4 protein [76], and the dynamics of transcription errors in DNA [76]. Parrondo's paradox has been studied in various interesting scenarios involving population genetics [39–41, 77].

In conventional sociobiology the standard dogma is that when chosing an optimal mate we are attracted to beauty, as those features we see in beauty are in fact indicators of a healthy mate—therefore to efficiently propagate our genes we seek attractive mates [78]. If this was really the complete picture, one might expect ugliness to have been selected out by now. The present theory does not seem to account for the common fact that two ugly parents can often produce an attractive child. Perhaps there is a Parrondian payoff in being a 'loser' at the dating game, where survival of the weakest can come into play. Arizmendi [79] has recently proposed a Parrondian dating model that fosters survival of the 'loser.' Satinover and Sornette [80, 81] propose a Parrondian model where they show that short term optimization can turn a positive expected gain into a negative one, thus showing in some circumstances it can pay to be the uncompetitive 'loser' who sticks to the minority.

In terms of the stockmarket, Boman [82, 83] has used a Parrondian game framework as a toy model for studying the dynamics of insider information. A number of Parrondo-like toy models for switching between poor performing investments are well-known. For example, Maslov and Zhang demonstrate a model where switching between volatile assets and non-performing cash reserves produces an increase in the gains [44] in a fashion not too dissimilar from Luenberger's *volatility pumping* method [48]. There are also closely related models such as the *excess growth* model of Fernholz and Shay [84] and Tom Cover's *universal portfolio* [85]. There are open questions yet to be explored in systemizing these effects and drawing out the exact connections with the large body of literature on Parrondo's paradox.

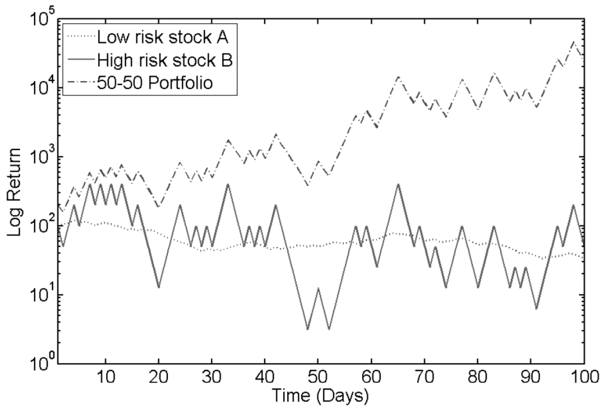

Fig. 9 Volatility pumping a low-risk stock with a high-risk stock. The dotted curve simulates a mediocre low-risk stock that in the long run neither wins nor loses. The solid curve represents a volatile stock that gives a 25% expected return, though is high-risk—a simple toy model of volatility is implemented here, where the stock simply halves or doubles, at random, its previous value at each time-step. The chained curve is found by selling both stocks at the end of each time step, adding the total cash to get $\$T$, then repurchasing them at the beginning of each time-step at a 50:50 split—that is, we purchase $\$T/2$ worth of the high-risk stock and $\$T/2$ worth of the low-risk-stock. This is process called *portfolio rebalancing.* Surprisingly, the chained curve grows exponentially, even though the two stocks individually do not perform as well. Both stocks start at Day 1 priced at \$100, and thus the combined portfolio (chained curved) starts at \$200. The vertical axis is the return in dollars plotted on a logarithmic scale. The return on the rebalanced portfolio is so large that we would not be able to see the individual curves, without the logarithmic plot.

Here, we focus on Luenberger's volatility pumping as it is a simple toy model that rather nicely illustrates the principles of 'winning' with poorer stocks in a clear way. Figure 9 simulates two stocks: one stock is stable but is mediocre and in the long run neither wins or loses significantly, the other stock has some growth but is volatile. A very simple toy model of volatility is used here, namely, that we randomly halve or double the stock value from day to day. Leunberger's method is to then sell both stocks each day and rebuy them implementing *portfolio rebalancing.* The rebalance operation is to take your total cash $\$T$, and buy $\$T/2$ of one stock and $\$T/2$ of the other—thus we maintain a 50:50 portfolio. This operation is repeated each day and remarkably it produces the top curve in Figure 9 with exponential growth. The simple Matlab code for producing this graph is in Appendix A. Now, it should be noted this is a stripped down toy model to illustrate the key idea that rebalancing

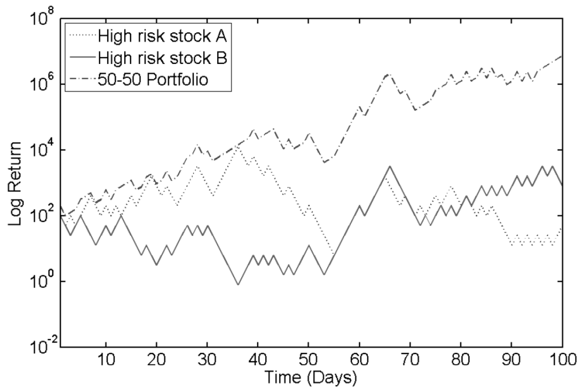

Fig. 10 Volatility pumping a high-risk stock with a high-risk stock. The scheme is identical to that in Fig. 9, with the exception that now both originating stocks are volatile. In this simulation both are generated by random halving and doubling, but are generated independently. Surprisingly, not only does the exponential growth still occur, but the winnings are about a factor of 100 higher than in the previous case.

creates growth. Surprisingly, Figure 10 shows the process still works even when both stocks are volatile. Of course, in reality one would not buy and sell everyday as transaction costs would be prohibitive. Also the halving and doubling operation is an artificial construct that is cleverly designed so that we never hit rock bottom—in reality, hitting the $y = 0$ axis is always the problem. However, the beauty of any toy model is it enables us to explore the pertinent features of an interesting effect. The open questions are why does volatility pumping work and how should the portfolio rebalancing strategy be optimized for best performance in a real scenario? Whilst both these questions are still being actively debated [86], from the point of view of ratchet science, volatility pumping must clearly be the result of an asymmetry that rectifies fluctuations in the market. This is the principle behind every Brownian ratchet and volatility pumping is no exception. The action of maintaining the 50:50 portfolio split guarantees that we are always buying low and selling high—recalling Section 3.2 we argued that this is indeed a ratcheting asymmetry.

5 Thermodynamics of Games of Chance

The fact the Parrondo's original games can be exactly derived via discretization of the Fokker-Planck equation [50–52] is of fundamental interest because it can then serve as a useful toy model for investigating the discrete-continuous interface. There is much emerging interest in the so-called discrete-continuous interface due to its importance in optimization and control problems [87]. Furthermore, via adopting a $t \to it$ Wick rotation, is possible to transform the Fokker-Planck equation into a Schrödinger equation [88] and this opens up interesting directions for future research of both Parrondo's games and the discrete-continuous interface in both classical and quantum regimes.

As Parrondo's games are deeply rooted in the thermodynamics of Brownian ratchets, via the Fokker-Planck equation, this has enabled the use of thermodynamic analogies when understanding the operation of the games. Various authors [32, 76, 89] have suggested simple analogies between Parrondo's games and physical Brownian motors. Amengual *et al.* [90] have taken this a step further and have proposed how to characterize Parrondo's games in terms of a thermodynamic engine efficiency.

This raises the open question of whether a generalized thermodynamic picture can be applied to arbitrary games of chance. To this end, in about 2001, I performed the following simple thought experiment. I imagined two players flipping a simple unbiased coin. Player 1 wins $1, given heads, and Player 2 wins $1, given tails. The coin is unbiased, which means that if I take a video of the game, and run the movie backwards we would not be able to tell the difference. Therefore the system displays *time-reversibility*, which is what we expect of physical systems that are in thermal equilibrium. Now, let us imagine we have a biased coin such that, say, Player 1 progressively wins. The situation is no longer time-reversible, because the forward and backward movies are now clearly different—in the forward movie Player 1 appears to win, but in the reverse movie Player 2 appears to win. This time-irreversible situation corresponds to physical systems that are out of equilibrium. Thus the situation of *detailed balance* when the coin is unbiased is analogous to thermodynamic equilibrium, and a biased coin is rather like a system out of equilibrium. Initially, this thought experiment appeared trivial and not particularly useful, however, in the next section we put it to good use illustrating a significant link between random manipulation of integer sequences and thermodynamics.

For those readers who are new to the concept that time-reversibility relates to thermal equilibrium, I recommend to consider the simple mental picture of Feynman's ratchet and pawl in Fig. 4. The two ends of the system are at different temperatures and the ratchet wheel rotates in one direction. Now if you make the two ends the same temperature, there is no net rotation of the ratchet wheel. Now take a movie of each of these two cases, and run the movie backwards. What do you see?

- Case 1: When the temperatures are different, $T_1 \neq T_2$. Here the backwards movie looks different to the forwards movie. Because the wheel is turning in a particular direction, the backwards movie would have the wheel turning the opposite way. As the movies are different, this is the acid test that tells us the process is irreversible.
- Case 2: When the temperatures are same, $T_1 = T_2$. Here we have thermal equilibrium, and so there is no net energy coming into the system and therefore the wheel does not rotate in a net direction. It just randomly jiggles back and forth, and no work is done. We run the movie backwards and now we cannot tell the difference—the backwards movie just looks like random jiggles, as does the forwards movie. Thus this case is reversible.

Thus, in summary, in thermal equilibrium we have reversibility, when out of thermal equilibrium we have irreversibility. This is a well-known principle in thermodynamics, but we have introduced the specific ratchet example as a nice physical picture for visualizing this concept.

6 A New Parrondian Effect: The Allison Mixture

A new form of Parrondo's paradox, namely the *Allison mixture*, has recently been reported [9]. Let us imagine two sequences of random numbers; we shall call them Sequence 1 and Sequence 2—they are totally random in that they are independent and have zero autocorrelation. For simplicity, we can consider these sequences to be random strings of 1s and 0s—however, note that the effect I am about to describe is not limited to binary sequences but is in fact general. If we now randomly scramble these two sequences to generate a third new sequence, naively we would expect this resulting sequence to also be completely random. It turns out that this is not always the case: counterintuitively the final sequence can have a finite autocorrelation ρ even though the ρ's of originating sequences are zero—this is what we call an *Allison mixture.*

Let us now be a little bit more precise about how we actually scramble the two sequences, so we can then write out an analytical expression for ρ to show that it can be in fact non-zero. We start at an arbitrary nth position of Sequence 1. We either move to position $n+1$ of Sequence 1 with probability $1-\alpha_1$ or skip to position $n+1$ of Sequence 2 with probability α_1. Whenever we find ourselves in Sequence 2, we hop to the next location on Sequence 1 with probability α_2 or advance one step within Sequence 2 with probability $1-\alpha_2$. We continue hopping back and forth between the two sequences in this manner and each digit that we sequentially land on is called out to form the new sequence. In this way a third new sequence is generated from the original two sequences by random hopping, using separate transition probabilities α_1 and α_2 to keep everything perfectly general. For the sake of further generality, let the means of the two originating sequences be μ_1 and μ_2.

It has been shown that the autocorrelation ρ for the generated sequence is [9],

$$\rho = \frac{1}{\sigma^2}\frac{\alpha_2}{\alpha_1+\alpha_2}(\mu_1-\mu_2)^2(1-\alpha_1-\alpha_2) \qquad (4)$$

where σ^2 is the variance of the final sequence. The full expression for the variance has been previously reported [9], but is not given here as it is not relevant for the following physical discussion. Our naive expectation is that a random mixture of random sequences should always result in $\rho = 0$—however, Equation 4 reveals that ρ is only zero provided $\mu_1 = \mu_2$ or $\alpha_1+\alpha_2 = 1$. If we break both these conditions, then we can legally produce a sequence with a non-zero ρ. The mathematics dictates to us that this must be the case, but the question is why? And what is the physical picture and basis for what is going on?

The previous section on the thermodynamics of chance—Section 5—contains many of the necessary clues to unravel the physical picture. Firstly, let us address physically why to get $\rho \neq 0$ we must have $\mu_1 \neq \mu_2$—the implication is that the means of the sequences are analogous to the temperature of a physical process. Loosely speaking, temperature is some measure that is proportional to the average of all the jiggling within a solid object. In the case of the random sequences, μ_1 and μ_2 are the averages of all the jiggling or varying numbers and play the same role as temperature. Thus when $\mu_1 \neq \mu_2$, we have an irreversible situation—the sequences are irreversibly mixed and we therefore get autocorrelation in the final sequences, because there is information loss. Recall that the originating sequences are random, and thus are incompressible in the Chaitin-Kolmogorov sense and thus contain maximal information in the Shannon sense. Thus by subjecting them to an irreversible process we know from thermodynamics that we must *lose* information, and thus redundancy must have crept into the final sequence leading to $\rho \neq 0$. Now, in the special case, when $\mu_1 = \mu_2$, we have reversible mixing, because this is analogous to thermal equilibrium where $T_1 = T_2$. If the process is reversible then there is no information loss, no redundancy is added, and therefore $\rho = 0$.

However, this is only part of the picture, as Equation 4 also predicts that to obtain a case where $\rho \neq 0$, we must also observe the $\alpha_1 + \alpha_2 \neq 1$ condition. So what is the physical reason why $\alpha_1 + \alpha_2 \neq 1$ is required to get non-zero ρ in the final sequence? To unravel the mystery we draw the state diagrams to illustrate the mechanism. Figure 11 illustrates the case when $\alpha_1 + \alpha_2 \neq 1$—the caption explains why switching between the two states leads to *memory persistence* that causes correlation in the final sequence (or anticorrelation in the case of antipersistence). Now for the case when $\alpha_1 + \alpha_2 = 1$, Figure 12 illustrates we get detailed balance between the probability of entering a state and the probability of staying in a state. (Note that staying within a state is also called a *self-transition*). The detailed balance implies there is no memory persistence and hence $\rho = 0$. An alternative valid explanation is to use the argument of Section 5 that explains why detailed balance implies a reversible process. So essentially we must have $\alpha_1 + \alpha_2 \neq 1$ to ensure irreversibility, which is a necessary condition for obtaining $\rho \neq 0$.

So now we begin to see the connection between an Allison mixture and Parrondian effects that require an asymmetry to interact with random behavior. Figure 12 is the symmetric case where we have detailed balance, and Figure 11 is the asymmetric case where detailed balance is broken. Symmetry breaking is the essence of all Parrondian and Brownian ratchet phenomena. The $\mu_1 \neq \mu_2$ condition is analogous to the $T_1 \neq T_2$ condition that is required for Feynman's ratchet of Figure 4 to operate. The $\alpha_1 + \alpha_2 \neq 1$ condition is analogous to the ratcheting mechanism in Figure 4 as these both are the sources of asymmetry. A number of open questions remain concerning the mathematical features of this switched system that can be thought of as a two-state discrete-time hidden Markov model—for example, Figure 13 illustrates that as $\alpha_1 \to 0$ and $\alpha_2 \to 0$ the direction of the limits intriguingly affect the final value of ρ. Another open question is that of a possible application for Allison mixtures—this remains to be seen, but possible areas of promise might be in encryption and in optimizing file compression. Another open

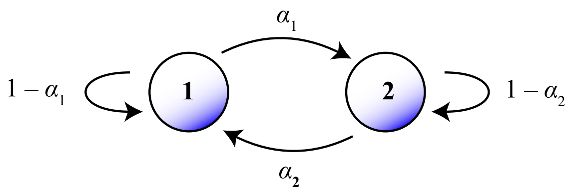

Fig. 11 State diagram for unbalanced switching process, when $\alpha_1 + \alpha_2 \neq 1$, giving rise to persistence. Circle 1 represents the state of landing in random Sequence 1 and Circle 2 represents the state we are in when we land on random Sequence 2. We jump between these two sequences to generate a new sequence. The apparent paradox of this Allison mixture is that the resulting sequence has non-zero autocorrelation even when the originating sequences have zero autocorrelation. Here, α_1 and α_2 are the transition probabilities of jumping between the two sequences. Notice, for example, as $\alpha_1 \to 0$ the probability of a self-transition to stay in Sequence 1 is high—so, if we are already in Sequence 1 we are likely to stay there. This can be thought of as a type of 'memory' of the system, which causes the new sequence to have non-zero autocorrelation. Note: this is *not* a form of memory in the sense that requires storage of a previous state—as there is no clear terminology, in the literature, for our probabilistic type of memory effect, we hereby call it *memory persistence*.

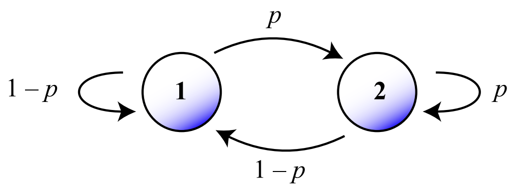

Fig. 12 State diagram for balanced switching process, when $\alpha_1 + \alpha_2 = 1$, resulting in no persistence. For simplicity we have inserted $p = \alpha_1 = 1 - \alpha_2$, which clearly reveals that the probability of entering a state exactly balances the probability of self-transition in that state. This detailed balance implies there is no memory persistence effect. Hence, the new generated sequence also has zero autocorrelation.

question to ask is if there are any links between Allison mixtures and biological evolution or genetics? Could it be that the redundancy that appears in sequences of non-coding (or 'junk') DNA are the result of something along the lines of Allison mixing (i.e. ratcheted random mixing)? In the case of coding DNA, random mutations are a biased process—for example frame shift mutations in DNA are more likely to occur in sequences with runs of a single base and some single base mutations are more probable. This together with the process of selection, which again is random but with biases, results in order that is created in a set of DNA sequences. These sequences encapsulate in an ordered way information about the regularities of the organism in its environmental context.

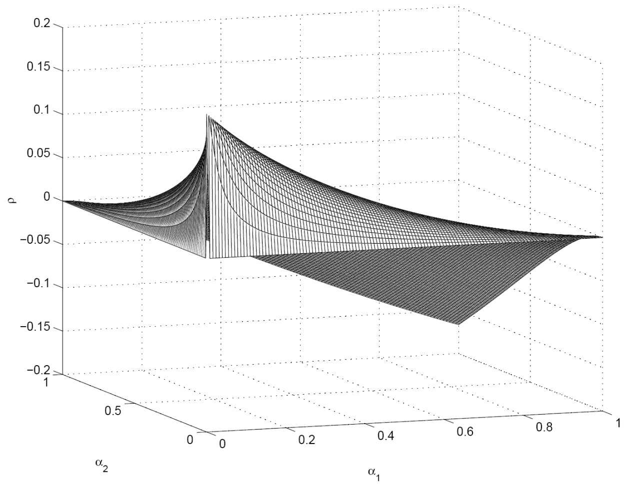

Fig. 13 Plot of the autocorrelation ρ of the generated sequence as a function of α_1 and α_2. In this specific example, the two originating binary sequences have $\mu_1 = 0.2$ and $\mu_2 = 0.6$. Pearce has named the peak of this graph the *pinnacle* [91]. The plot shows that the system displays some mathematically curious features. Surprisingly, $\rho = 0$ does not occur at a unique point in the parameter space. Another intriguing feature is that as $\alpha_1 \to 0$ and $\alpha_2 \to 0$, whether ρ ends up at zero or at Pearce's pinnacle depends on the direction that you approach the limits.

7 Conclusion

There are two key take home messages that the study of Parrondo effects reveal: (i) the process of switching is a nonlinearity and therefore switching can radically alter the overall system behavior, and (ii) the interaction between noise and an asymmetry can give rise to directed motion even against a gradient, provided we are out of equilibrium.

Physicists have traditionally sought symmetry in Nature—a new challenge for future research is to now search for asymmetries and observe how they interact with noise or random behavior. On a more philosophical note, we might pose the question "Should we consider noise or randomness as a special form of order?" As well as our discussion on Parrondo effects, there are other examples that point to this: (i) a random walk displays self-similarity, (ii) randomly switched processes can produce fractals, (iii) according to Shannon, noise packs in maximal information, and (iv) as Chaitin points out, even the integers have noisy properties [92]. After all, noise is the most ordered way to avoid redundancy. There are many situations where noise appears to give rise to order and the challenge is to identify the general mathematical principles behind this.

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Appendix A

The simple Matlab code for the demonstration of the principle of volatility pumping a high-risk stock together with a low-risk stock, is as follows:

```
  % high-low volatility pumping
clc; clear;
day = 100;
xscale = [1:day];

  % low risk stock initialization
A1(1) = 100;
A2(1) = 100;

  % high risk stock initialization
B1(1) = 100;
B2(1) = 100;

  % random halving and doubling
dh = ceil(2.*rand(day,1));
idx = find(dh==1);
dh(idx) = 0.5;
R = (rand(day,1) - 0.5)./5;

  % portfolio management
for ii=2:day
T(ii-1) = A2(ii-1) + B2(ii-1);
A2(ii) = T(ii-1)/2 + T(ii-1)/2*R(ii-1);
B2(ii) = T(ii-1)/2*dh(ii-1);
A1(ii) = A1(ii-1) + A1(ii-1)*R(ii-1);
B1(ii) = B1(ii-1)*dh(ii-1);
end
T(day) = A2(day) + B2(day);

  % plot graph
figure;
hx = plot(xscale, A1, xscale, B1, xscale, T);
set(hx(1),'linewidth',2,'linestyle',':');
set(hx(2),'linewidth',2,'linestyle','-');
```

```
set(hx(3),'linewidth',2,'linestyle','-.');
set(gca,'linewidth',2,'FontName','Arial',
'FontSize',20,'xlim',[1 100],'yscale','log'); box on;
xlabel('Time (Days)'); ylabel('Log Return');
legend('Low risk stock A','High risk stock B',
'50-50 Portfolio','Location','NorthWest');
```

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